# Discussion 16 Worksheet

## Conservative vector fields and Green's theorem

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MATH 53 Multivariable Calculus

## 1 Conservative Vector Fields

For each of the following vector fields $\vec{F}$, either prove that $\vec{F}$ is conservative by finding a function $f$ such that $\nabla f = \vec{F}$, or prove that $f$ is not conservative. We can now do this systematically as showed in lecture.

1. $\vec{F}(x, y) = (xy + y^2)\,\vec{i} + (x^2 + 2xy)\,\vec{j}$
2. $\vec{F}(x, y) = y^2 e^{xy}\,\vec{i} + (1 + xy)e^{xy}\,\vec{j}$
3. $\vec{F}(x, y, z) = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$

## 2 Apply FTL

Find a function $f$ such that $\vec{F} = \nabla f$ and use that to evaluate $\int_C \vec{F} \circ d\vec{r}$ along the curve $C$.

1. $\vec{F}(x, y) = x^2y^3\,\vec{i} + x^3y^2\,\vec{j}$ and $C : \vec{r}(t) = \langle t^3 - 2t, t^3 + 2t\rangle, 0 \leq t \leq 1$.
2. $\vec{F}(x, y) = \langle yz, xz, xy + 2z\rangle$ and $C$ is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$.
3. $\vec{F}(x, y, z) = \langle yze^{xz}, e^{xz}, xye^{xz}$ and $C : \vec{r}(t) = \langle t^2 + 1, t^2 - 1, t^2 - 2t\rangle, 0 \leq t \leq 2$.

## 3 More on line integrals

1. Show that the line integral is independent of path and evaluate it. $\int_C 2xe^{-y}dx+(2y-x^2e^{-y})dy$ and $C$ is any path from $(1, 0)$ to $(2, 1)$.
2. Find the work done by the force field $\vec{F} = \langle x^3, y^3\rangle$ in moving an object from $(1, 0)$ to $(2, 2)$.

## 4 Challenge

1. Consider the vector field
$$\vec{F}(x, y) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$
First show that $\partial P/\partial y = \partial Q/\partial x$. Then show that $\int_C \vec{F} \circ d\vec{r}$ is not independent of path. Does this contradict what we saw in lecture?

## 5 Line Integrals to Double Integrals

Use Green's theorem to convert each of the following line integrals $\int_C \vec{F} \circ d\vec{r}$ to double integrals. Then evaluate. All curves $C$ are oriented counterclockwise.

1. $C$ is the ellipse $x^2 + y^2/4 = 1$ and $\vec{F}(x, y) = \langle 2x - y, 3x + 2y \rangle$.
2. $C$ is the circle $x^2 + y^2 = 1$ and $\vec{F}(x, y) = \frac{1}{3}\langle -y^3, x^3 \rangle$.
3. $C$ is the triangle with vertices at $(0, 0), (1, 0), (0, 1)$ and $\vec{F}(x, y) = \langle x^2 y, e^{y^2} + x \rangle$.

## 6 More on Green's Theorem

1. Let $C$ be a simple, positively oriented, closed curve in $\mathbb{R}^2$. Using Green's theorem, check that $\int_C f(x)dx + g(y)dy = 0$ for arbitrary smooth functions $f, g$. Can you give an explanation without Green's theorem?

2. Consider the non-standard parameterization of the unit circle $x = \sin(t), y = \cos(t)$ with $0 \le t \le 2\pi$. Check that $\int_C x dy$ is not the area enclosed by $C$, as "promised" by Green's Theorem. What went wrong?

3. Let $C$ be the square centered at the origin with side length 4, oriented counterclockwise. Compute $\int_C \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$.

   Hint: Recall that the vector field $\langle P, Q \rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$ satisfies $P_y = Q_x$ but is not conservative because $\int_\gamma P dx + Q dy = 2\pi$ where $\gamma$ is the unit circle, oriented counterclockwise.

4. Let $\vec{F}$ be the vector field in the previous problem. Explain why, using Green's theorem, if $C$ is a simple positively oriented curve contained in the upper half plane $y > 0$, then $\int_C \vec{F} \circ d\vec{r} = 0$.

## 7 True/False

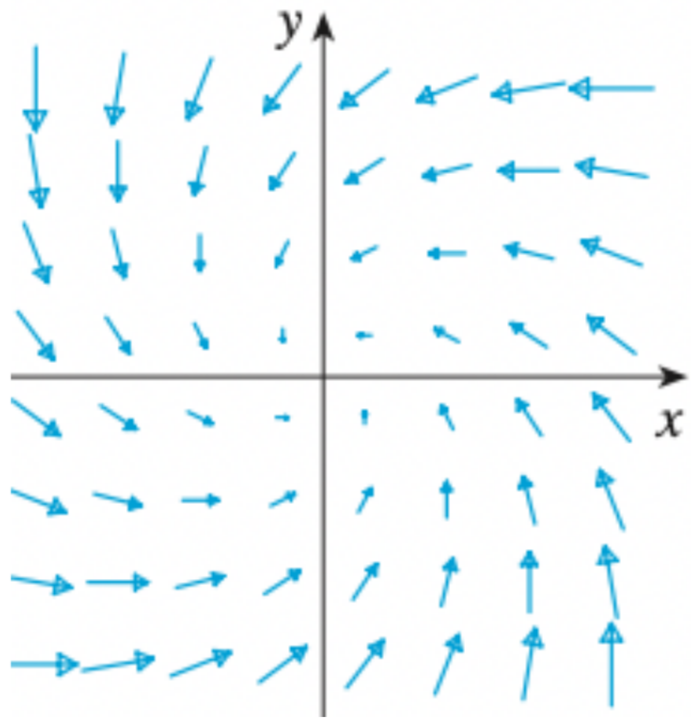
(a) T F If the vector field $\vec{F} = \langle P, Q, R \rangle$ is conservative and the components have continuous first-order partial derivatives then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

(b) T F You're asked to find the curve that requires the least work for the force field $\vec{F}$ to a move a particle from point to another point. You find that $\vec{F}$ is conservative. Then there a unique path satisfying this request.

(c) T F The line integral $\int_C y dx + x dy + xyz dz$ is path independent.

(d) T F The following vector field is conservative.




**Note:** These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.